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A CLASS OF CONSTRUCTIONS FOR TURÁN'S (3, 4)-PROBLEM

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Let $f(n)$ denote the minimal number of edges of a 3-uniform hypergraph $G=(V,E)$ on n vertices such that for every quadruple $Y\subset V$ there exists $Y\supset e\in E$. Turán conjectured that $f(3k)=k(k-1)(2k-1)$. We prove that if Turán's conjecture is correct then there exist at least 2^{k-2} non-isomorphic extremal hypergraphs on $3k$ vertices.

1. Introduction

A 3-uniform hipergraph $G=(V,E)$ is called a (3, 4)-*graph* if for any 4-element set $Y\subset V$ there is an edge $e\in E$ such that $e\subset Y$. In other words, if there is no empty 4-set in G.

We call the following problem "Turán's (3, 4)-problem" [2]. Find the minimal number $f(n)$ of edges of a (3, 4)-graph on n vertices.

The following 3-graph is a (3, 4)-graph:

(a) $V=V_1\cup V_2\cup V_3$ is a partition of V. $|V_1|=\left[\frac{n}{3}\right]$, $|V_2|=\left[\frac{n+1}{3}\right]$, $|V_3|=\left[\frac{n+2}{3}\right]$.

$\{u,v,w\}\in E$ iff u, v and w are different and one of the following conditions is satisfied:

(b) $\{u,v,w\}\subset V_j$ for some $1\leqq j\leqq 3$,

(c) $\{u,v\}\subset V_j$, $w\in V_{j+1}$ for some $1\leqq j\leqq 3$.

Here and in the whole paper j is understood mod 3. From this construction it follows that $f(n)\leqq\varphi(n)$ where

$$\varphi(n)=\begin{cases}(2k-1)(k-1)\cdot k, & \text{if } n=3k;\\ (2k-1)\cdot k^2, & \text{if } n=3k+1;\\ (2k+1)\cdot k^2, & \text{if } n=3k+2.\end{cases}$$

Turán conjectured $f(n)=\varphi(n)$. Brown [1] has shown that in case of $n\equiv 0$ (mod 3) there are at least $n/3-1$ non-isomorphic (3, 4)-graphs with n vertices and $\varphi(n)$ edges. In the present paper we construct $2^{n/3-2}$ such graphs. This fact indicates that if Turán's conjecture is correct, it is not easy to prove it.

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2. The construction

Let $n=3k$, m, l_1, l_2, ..., l_m be positive integers satisfing $l_m \geqq 2$ and $\sum_{i=1}^{m} l_i = k$. The 3-uniform hypergraph $G_m(l_1, l_2, \ldots, l_m)=(V, E)$ defined in the following way.

$$V = V_1 \cup V_2 \cup V_3,$$

$$V_j = \bigcup_{i=1}^{m} V_{ij}$$

are partitions of V and V_j, respectively.

$$|V_{ij}| = l_i \quad (1 \leqq j \leqq 3,\ 1 \leqq i \leqq m).$$

The triple $\{u, v, w\}$ belongs to E iff u, v and w are different and one of the following conditions is satisfied:

(i) $$u \in V_{i_1, j}, \quad v \in V_{i_2, j}, \quad w \in V_{i_3, j},$$

for some $j \in \{1, 2, 3\}$ and $1 \leqq i_1 \leqq i_2 \leqq i_3 \leqq m$ such that $i_1 \equiv i_2 \pmod 2$;

(ii) $$u \in V_{i_1, j}, \quad v \in V_{i_2, j}, \quad w \in V_{j+1},$$

for some $j \in \{1, 2, 3\}$ and $1 \leqq i_1 \leqq i_2 \leqq m$ such that i_1 is odd;

(iii) $$u \in V_{i_1, j}, \quad v \in V_{i_2, j}, \quad w \in V_{j-1},$$

for some $j \in \{1, 2, 3\}$ and $1 \leqq i_1 \leqq i_2 \leqq m$ such that i_1 is even;

(iv) $$u \in V_{i_1, j}, \quad v \in V_{i_2, j}, \quad w \in V_{i_3, j-1},$$

for some $j \in \{1, 2, 3\}$ and $1 \leqq i_3 < i_1 \leqq i_2 \leqq m$ such that i_1 is odd, i_3 is even;

(v) $$u \in V_{i_1, j}, \quad v \in V_{i_2, j}, \quad w \in V_{i_3, j+1},$$

for some $j \in \{1, 2, 3\}$ and $1 \leqq i_3 < i_1 \leqq i_2 \leqq m$ such that i_1 is even, i_3 is odd.

It is easy to check that $G_1(k)$ is the construction described in the introduction and $G_2(l_1, l_2)$ is Brown's (3, 4)-graph [1]. If it does not cause any ambiguity $G_m(l_1, l_2, \ldots, l_m)$ will be denoted by G_m.

3. Properties of G_m

Proposition 1. *For any pair $\{u, v\} \subset V_{m,j}$ $u \neq v$ there are precisely $2k-2$ edges e satisfying $\{u, v\} \subset e \in E$.*

Proof. *Case 1.* m is odd. The third vertex of the edge $\{u, v, w\}$ belongs to V_j iff $w \in V_{1,j} \cup V_{3,j} \cup \ldots \cup V_{m,j} \setminus \{u, v\}$ (condition (i)). This means that $\{u, v\}$ belongs to $l_1 + l_3 + \ldots + l_m - 2$ edges being entirely in V_j. It follows from condition (ii) that all the k edges satisfying $w \in V_{j+1}$ belong to G_m. According to (iv), the third w can be chosen $l_2 + l_4 + \ldots + l_{m-1}$ ways from V_{j-1}. There are no edges containing $\{u, v\}$

and satisfying (iii) or (v). Consequently, the number of edges of G_m containing $\{u, v\}$ is

$$(l_1+l_3+\ldots+l_m-2)+k+(l_2+l_4+\ldots+l_{m-1}) = 2k-2.$$

Case 2. m is even. Analogously to Case 1, there are $l_2+l_4+\ldots+l_m-2$ edges in G_m, containing $\{u, v\}$ is in V_j. We can choose w in k different ways from V_{j-1} and $l_1+l_3+\ldots+l_{m-1}$ ways from V_{j+1} (conditions (iii) and (v) resp.). There are no edges containing $\{u, v\}$ and satisfying (ii) or (iv). The number of considered edges is the same as above. ∎

If G is a 3-graph $V(G)$ and $E(G)$ denote its vertex set and edge set respectively.

Let us notice that our notations V_{ij} are somewhat inconsequent in the construction: $V_{ij}=V_{ij}(l_1, l_2, \ldots, l_m)$ can depend on the parameters $l_1, l_2, \ldots, l_m$. In what follows we will use induction, properties of $G_m(l_1, l_2, \ldots, l_m)$ will be proved by using $G_{m-1}(l_1, l_2, \ldots, l_{m-2}, l_{m-1}+l_m)$ on the same vertex set. Therefore V_{ij} will be used for $V_{ij}(l_1, l_2, \ldots, l_m)$ and it will be supposed that $V_{ij}(l_1, l_2, \ldots, l_{m-2}, l_{m-1}+l_m)= =V_{ij}$ if $1\leqq i\leqq m-2$, $V_{m-1,j}(l_1, l_2, \ldots, l_{m-2}, l_{m-1}+l_m)=V_{m-1,j}\cup V_{m,j}$ for every $1\leqq j\leqq 3$.

Proposition 2. *$G_m(l_1, l_2, \ldots, l_m)$ and $G_{m-1}(l_1, l_2, \ldots, l_{m-2}, l_{m-1}+l_m)$ may differ only in edges being at least 2 vertices from $V_{m,j}$ for some $1\leqq j\leqq 3$. More precisely*

$$\{e\in E(G_m(l_1, l_2, \ldots, l_m))\,|\,|e\cap V_{m,j}| \leqq 1, \quad \text{for} \quad 1 \leqq j \leqq 3\} =$$
$$= \{e\in E(G_{m-1}(l_1, l_2, \ldots, l_{m-2}, l_{m-1}+l_m))\,|\,|e\cap V_{m,j}| \leqq 1, \quad \text{for} \quad 1 \leqq j \leqq 3\}.$$

Proof. Suppose that $e\in E(G_m(l_1, l_2, \ldots, l_m))$ and e is of type (i). If $i_2<m$ then e is obviously in $E(G_{m-1}(l_1, l_2, \ldots, l_{m-2}, l_{m-1}+l_m))$. If $i_2=m$ then $i_3=m$, e has two vertices in $V_{m,j}$. Suppose now that $e\in E(G_m(l_1, l_2, \ldots, l_m))$ of type (ii)—(v). If $i_1<m$ then e belongs to $E(G_{m-1}(l_1, l_2, \ldots, l_{m-2}, l_{m-1}+l_m))$ otherwise $i_1=i_2=m$.

The other direction can be proved analogously. ∎

Proposition 3. *$|E(G_m(l_1, l_2, \ldots, l_m))|=|E(G_1(k))|=\varphi(3k)$ holds for any set m, $l_1, l_2, \ldots, l_m, k$ of natural numbers satisfying $l_m\geqq 2$, $\sum_{i=1}^{m} l_i=k$.*

Proof. We use induction on m. The statement is trivial for $m=1$. Let us suppose that it is true for any $m'<m$.

$$|\{e\in E(G_m(l_1, l_2, \ldots, l_m))\,|\,|e\cap V_{m,j}| \leqq 1 \quad \text{for} \quad 1 \leqq j \leqq 3\}| =$$
$$= |\{e\in E(G_{m-1}(l_1, l_2, \ldots, l_{m-2}, l_{m-1}+l_m))\,|\,|e\cap V_{m,j}| \leqq 1 \quad \text{for} \quad 1 \leqq j \leqq 3\}|$$

follows from Proposition 2. However, by Proposition 1, the number of edges containing a pair $\{u, v\}\subset V_{m,j}$ is $2k-2$ both in $G_m(l_1, l_2, \ldots, l_m)$ and in $G_{m-1}(l_1, l_2, \ldots$ $\ldots, l_{m-2}, l_{m-1}+l_m)$. Finally, both graphs contain all the triples being completely in $V_{m,j}$. Consequently, we have

$$|E(G_m(l_1, l_2, \ldots, l_m))| = |E(G_{m-1}(l_1, l_2, \ldots, l_{m-2}, l_{m-1}+l_m))| = |E(G_1(k))|. \quad ∎$$

Proposition 4. *$G_m(l_1, l_2, \ldots, l_m)$ is a (3, 4)-graph.*

Proof. We use induction, again. For $m=1$ the statement is known. Suppose, that it is true for any $m'<m$ and prove it for m.

Let $Y=\{v_1, v_2, v_3, v_4\}$ be a quadruple containing no edge in $G_m(l_1, l_2, \ldots, l_m)$. If $|V_{m,j}\cap Y|\leqq 1$ for all $j\in\{1, 2, 3\}$ then Proposition 2 implies that Y contains no edge in $G_{m-1}(l_1, l_2, \ldots, l_{m-2}, l_{m-1}+l_m)$. This contradicts our inductional hypothesis. Therefore one of $V_{m,j}$ contains at least two elements of Y, say $\{v_1, v_2\}\subset V_{m,1}$. We distinquish two cases according to the parity of m.

Case 1. m is even. It follows from (iii) that $Y\cap V_3=\emptyset$.

Suppose that $|V_1\cap Y|\geqq 3$, that is, there is an i_1 such that $v_3\in V_{i_1,1}$. If i_1 is even then $\{v_1, v_2, v_3\}\in E(G_m)$ holds by (i), contradicting our supposition. If i_1 is odd then $v_4\in V_2$ leads to a contradiction by (ii). That is in this case $v_4\in V_{i_2,1}$ for some odd i_2. Now $\{v_3, v_4, v_1\}\in E(G_m)$ gives the contradiction by (i). We may conclude $Y\cap V_1=\{v_1; v_2\}$.

Therefore $v_3, v_4\in V_2$, that is, $v_3\in V_{i_3,2}$, $v_4\in V_{i_4,2}$ for some i_3, i_4. If one of them is odd then (v) gives a contradiction. If $i_3\equiv i_4\equiv 0 \pmod 2$ then $\{v_3, v_4, v_1\}\in \in E(G_m)$ follows by (iii).

Case 2. m is odd. The proof in this case is analogous to Case 1. ∎

Let us now investigate the complete subgraphs of G_m on $k+1$ vertices. Suppose that $M\subset V(G_m)$, $|M|=k+1$, and every three-element subset of M is in $E(G_m)$. As G_m contains no edge with exactly one vertex of each V_j, $M\cap V_j=\emptyset$ must hold for some j. $M\cap V_3=\emptyset$ may be supposed for definiteness.

$|M|>k$ implies that for some $1\leqq i\leqq m$ hold $M\cap V_{i,1}\neq\emptyset$ and $M\cap V_{i,2}\neq\emptyset$ simultaneously. Say $v\in V_{i,1}\cap M$, $u\in V_{i,2}\cap M$. If $w\in M\setminus\{u, v\}$ then $\{u, v, w\}\in \in E(G_m)$. We may conclude by (i)—(v) that w must be in the set

$$M_1 = (V_{1,1}\cup V_{3,1}\cup\ldots\cup V_{i,1})\cup\left(\bigcup_{s=i+1}^{m} V_{s,1}\right)\cup(V_{2,2}\cup V_{4,2}\cup\ldots\cup V_{i-1,2})\setminus\{v\}$$

if i is odd and in the set

$$M_2 = (V_{1,1}\cup V_{3,1}\cup\ldots\cup V_{i-1,1})\cup(V_{2,2}\cup V_{4,2}\cup\ldots\cup V_{i,2})\cup\left(\bigcup_{s=i+1}^{m} V_{s,2}\right)\setminus\{u\}$$

if i is even. The sets $M_1\cup\{u, v\}$ and $M_2\cup\{u, v\}$ have cardinalities $k+1$. They are the only candidates for M, containing $\{u, v\}$. However, if $i<m$ then $V_{m-1,1}\cup \cup V_{m,1}\subset M_1\cup\{u, v\}$ and $V_{m-1,2}\cup V_{m,2}\subset M_2\cup\{u, v\}$ but G_m contains no edge $\{x, y, z\}$ of form $\{x, y\}\subset V_{m,j}$, $z\in V_{m-1,j}$. Therefore i must be equal to m. The $(k+1)$-element sets M inducing a complete subgraph in G_m are

$$M = V_{1,j}\cup V_{2,j+1}\cup V_{3,j}\cup V_{4,j+1}\cup\ldots\cup V_{m,j}\cup\{u\} \qquad u\in V_{m,j+1}$$

if m is odd and

$$M = V_{1,j}\cup V_{2,j+1}\cup V_{3,j}\cup V_{4,j+1}\cup\ldots\cup V_{m,j+1}\cup\{v\} \quad v\in V_{m,j}$$

if m is even.

Their number is $3\cdot l_m$ in both cases. Moreover they can be divided into 3 classes of l_m elements in such a way that the intersection of the l_m complete subgraphs being in one class is a complete subgraph on k vertices. As the isomorphic picture of a complete 3-graph is a complete 3-graph we have proved the following Lemma:

Lemma 5. *If* $G_m(l_1^1, l_2^1, \dots, l_m^1)$ $(l_m^1 \geqq 2)$ *and* $G_r(l_1^2, l_2^2, \dots, l_r^2)$ $(l_r^2 \geqq 2)$ *are isomorphic then* $l_m^1 = l_r^2$ *and for any isomorphism* ψ *and index* $j \in \{1, 2, 3\}$ *there is a* $j' \in \{1, 2, 3\}$ *such that* $\psi(V_{m,j}(l_1^1, l_2^1, \dots, l_m^1)) = V_{r,j'}(l_1^2, l_2^2, \dots, l_r^2)$. ∎

The following lemma immediately follows by Proposition 2:

Lemma 6. *Let us fix the vertices* $v_1 \in V_{m,1}, v_2 \in V_{m,2}, v_3 \in V_{m,3}$ *of* $G_m(l_1, l_2, \dots, l_m)$. *Then the 3-graph induced in* $G_m(l_1, l_2, \dots, l_m)$ *by* $V \setminus \bigcup_{j=1}^{3} (V_{m,j} \setminus \{v_j\})$ *is isomorphic to* $G_{m-1}(l_1, l_2, \dots, l_{m-2}, l_{m-1}+1)$. ∎

Now we are able to prove

Proposition 7. *If* $G_m(l_1^1, l_2^1, \dots, l_m^1)$ *and* $G_r(l_1^2, l_2^2, \dots, l_r^2)$ *are isomorphic* $l_m^1 \geqq 2$, $l_r^2 \geqq 2$ *then* $m = r$, $l_1^1 = l_1^2$, $l_2^1 = l_2^2, \dots, l_m^1 = l_m^2$.

Proof. We use induction on $m+r$. If $m+r=2$ then the statement is trivial. Suppose $m+r>2$ and that the proposition is true for smaller values. Let ψ be an isomorphism between $G_m(l_1^1, l_2^1, \dots, l_m^1)$ and $G_r(l_1^2, l_2^2, \dots, l_r^2)$. Lemma 5 implies $l_m^1 = l_r^2$ and the existence for every $j \in \{1, 2, 3\}$ of $j' \in \{1, 2, 3\}$ such that

$$(*) \qquad \psi(V_{m,j}(l_1^1, l_2^1, \dots, l_m^1)) = V_{r,j'}(l_1^2, l_2^2, \dots, l_r^2).$$

Let us note that $l_m^1 = l_r^2 < k$ and $m>1, r>1$ follow from $m+r>2$. Fix the vertices $v_j \in V_{m,j}(l_1^1, l_2^1, \dots, l_m^1)$ $(1 \leqq j \leqq 3)$ and consider the graphs G^1 and G^2 induced by $V(G_m(l_1^1, l_2^1, \dots, l_m^1)) \setminus \bigcup_{j=1}^{3} (V_{m,j}(l_1^1, l_2^1, \dots, l_m^1) \setminus \{v_j\})$ and $V(G_r(l_1^2, l_2^2, \dots l_r^2)) \setminus \bigcup_{j'=1}^{3} (V_{r,j'}(l_1^2, l_2^2, \dots, l_r^2) \setminus \{\psi(v_j)\})$ in $G_m(l_1^1, l_2^1, \dots, l_m^1)$ and $G_r(l_1^2, l_2^2, \dots, l_r^2)$ respectively. The restriction of ψ on $V(G^1)$ gives an isomorphism between G^1 and G^2, by (*). We know from Lemma 6 that G^1 is isomorphic to $G_{m-1}(l_1^1, l_2^1, \dots, l_{m-2}^1, l_{m-1}^1+1)$ and G^2 is isomorphic to $G_{r-1}(l_1^2, l_2^2, \dots, l_{r-2}^2, l_{r-1}^2+1)$. It follows by the inductional hypothesis that $m-1 = r-1$, $l_1^1 = l_1^2$, $l_2^1 = l_2^2, \dots, l_{m-2}^1 = l_{r-2}^2$, $l_{m-1}^1 + 1 = l_{r-1}^2 + 1$. ∎

Proposition 8. *The number of* $(3, 4)$*-graphs with* $3k$ $(k \geqq 2)$ *vertices and* $\varphi(3k)$ *edges is at least* 2^{k-2}.

Proof. According to Proposition 7, the number of non-isomorphic graphs $G_m(l_1, l_2, \dots, l_m)$ satisfying $\sum_{i=1}^{m} l_i = k$ is equal to the number of solutions of the equation

$$l_1 + l_2 + \dots + l_{m-1} + \bar{l}_m = k-1,$$

where $\bar{l}_m = l_m - 1$, in natural numbers. This is known to be $\binom{k-2}{m-1}$. That is, the total number of non-isomorphic (3, 4)-graphs with $3k$ vertices and $\varphi(3k)$ edges is at least

$$\sum_{m=1}^{k-1} \binom{k-2}{m-1} = 2^{k-2}. \ ∎$$

Remark. Deleting any vertex of G_m a (3, 4)-graph with $3k-1$ vertices and $\varphi(3k-1)$ edges can be obtained. If two such vertices are deleted from G_m which belong to exactly $k-1$ edges then the obtained graph has $3k-2$ vertices and $\varphi(3k-2)$ edges. It means that our construction gives a lot of non-isomorphic $(3, 4)$-graphs with n vertices and $\varphi(n)$ edges for $n \not\equiv 0 \pmod 3$, too.

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